Let $\alpha = a \cos \theta_0$, $\beta = a \sin \theta_0$. Then

$$s - s_0 = a \sin (\theta - \theta_0).$$

But this is the intrinsic equation of the cycloid.[1]

BOOK REVIEWS.

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An Elementary Course in Synthetic Projective Geometry. By Derrick Norman Lehmer. Ginn and Company, Boston, 1917. xiii + 123 pages.

Lehmer's *Projective Geometry* "is intended to give in as simple a way as possible the essentials of synthetic projective geometry." The author has done this in the 74 pages of Chapters I—VI, VIII by the methods of pure geometry without the use of anharmonic ratios, circles or any other metric notions. It is his idea that "a purely projective notion ought not to be based on metrical foundations." "The course is not intended to furnish an illustration of how a subject may be developed from the smallest possible number of fundamental assumptions. The author is aware of the importance of work of this sort, but he does not believe it is possible at the present time to write a book along such lines which shall be of much use for elementary students." For the purpose of this course the student should have a thorough knowledge of high-school plane geometry and enough solid geometry to understand the proof of Desargues's theorem about perspective triangles: "*If two triangles ABC and $A'B'C'$ are so situated that the lines AA', BB', CC' all meet in a point, then the pairs of sides AB and $A'B'$, BC and $B'C'$, CA and $C'A'$ all meet on a straight line, and conversely.*"

The first chapter of the book is devoted to the important notion of one-to-one correspondence. In it is explained the need of the fiction about points and lines at infinity. On page 7, after some remarks on one-to-one correspondences being continuous, the author says: "In the case of point-rows this continuity is subject to exception in the neighborhood of the point 'at infinity.'" The reviewer doubts that the student will understand what the "neighborhood of the point at infinity" is.

The theorem of Desargues is very easily proved when the two triangles are not in one plane. When they are in one plane the usual method of proof consists in showing that a third triangle $A''B''C''$ can be found which is perspective to both ABC and $A'B'C'$ from two different centers of projection, and in twice making use of the theorem for two perspective triangles not in one plane. The only difficulty which the student is likely to have with this proof is in the making and the visualizing of the diagram. Lehmer proves the theorem for two triangles not in one plane in the usual way, and draws an illustrative figure;

[1] See Williamson, *loc. cit.*, p. 338.

then he disposes of the more difficult part of the theorem as follows: "If now we consider the figure a plane figure, the points P, Q, and R still all lie on a straight line, which proves the theorem." This appeal to the student's knowledge of perspective drawing really assumes that if ABC and $A'B'C'$ are two perspective triangles in a plane it is always possible to find two other perspective triangles not in one plane of which ABC and $A'B'C'$ are the perspective picture. When this is argued in detail, as it is in the eleventh edition of the "Encyclopedia Britannica," volume 22, page 428, in the article on "Projection," and a diagram is drawn to illustrate, the proof is just as difficult as the usual proof mentioned above. The author should have given a little more space to this important theorem. The proof is hardly convincing in its brief form. In this same chapter are the definitions of four harmonic points, lines and planes and also this definition of projectivity: "Two fundamental forms are projectively related to each other when a one-to-one correspondence exists between the elements of the two and when four harmonic elements of the one correspond to four harmonic elements of the other." One likes to think of the adjective *projective* in the phrase *two projective point-rows* as meaning that one point-row can be obtained from the other by a series of projections and sections; in fact in paragraph 9 in the first chapter our author has used the phrase *projectively related* in just this sense. But the definition of projectivity quoted above has its advantages. Reye uses it in his *Geometrie der Lage;* but later, to use his own words, he "justifies the use of the term *projective*" by proving "that two projective one-dimensional forms may always be considered as the first and last of a series of forms of which each is perspective both to the one preceding it and to the one following it." Lehmer's book does not contain this proof. Reye's proof is based on this fundamental theorem of projective geometry: "*If two projective point-rows, superposed upon the same straight line, have more than two self corresponding points, they must have an infinite number, and every point corresponds to itself; that is, the two point-rows are not essentially distinct.*" Lehmer proves this theorem in Chapter III by making use of the postulate of continuity in this form: "We now assume, explicitly, the fundamental postulate that the correspondence is *continuous*, that is, that *the distance between two points in one point-row may be made arbitrarily small by sufficiently diminishing the distance between the corresponding points in the other.*" In this same Chapter III a point-row of the second order is defined as the locus of the points of intersection of corresponding rays of two projective pencils; and there is a similar definition of a pencil of rays of the second order. Point-rows of the second order are studied in Chapter IV. It is there proved that such a curve is uniquely determined by any five of its points. Pascal's theorem is proved in its original form and also in the cases in which the inscribed hexagon has degenerated into a pentagon, a quadrangle, and a triangle. Enough construction problems are given to show the great power of the theorem. It is pointed out that circles and conic sections are point-rows of the second order and the statement is made that "it will appear later that a point-row of the second order is a conic section. In the future, therefore, we shall refer to the point-row of the second order as a conic."

Chapter V is a similar chapter on pencils of rays of the second order; the development is about the same as in Chapter IV with Brianchon's theorem taking the place of Pascal's. At the end of this chapter the author calls attention to the principle of duality as evidenced by a comparison of Chapters IV and V. In some books the principle of duality is only a working principle; its validity is not rigorously established but the method of translating from a theorem to its correlative is explained and the validity of the latter is established by translating the proof of the former. In such books it is quite the custom to exhibit the theorem and its correlative in parallel columns on the same page. In other books, for example in "Projective Geometry" by Veblen and Young, it is shown that the duality exists in the axioms on which projective geometry is based, and it is argued that the duality therefore exists in the theorems derivable from those axioms. Poncelet, the discoverer of the principle, established its validity by means of the theory of poles and polars. This is the method used by Lehmer in Chapter VI. The starting point of this chapter is the following pair of theorems: "If a quadrangle be inscribed in a conic, two pairs of opposite sides and the tangents at opposite vertices intersect in four points, all of which lie on a straight line." "If a quadrilateral be circumscribed about a conic, the lines joining two opposite points of contact and the lines joining two pairs of opposite vertices are four lines which meet in a point." Lehmer is not always careful to distinguish between quadrangle and quadrilateral. For instance, in the first of these two theorems he says quadrilateral when he means quadrangle.

An involution of points on a line is often defined as the special case of two superposed point-rows in which every point on the line has the same correspondent whether it is thought of as belonging to the one point-row or the other. In Chapter VIII Lehmer introduces the student to the study of involution in a simpler way by first defining, in terms of a quadrangle and a transversal, what is meant by saying that three pairs of points on a line are in involution and by using this definition to show what is meant by saying that all the points of a line are in involution. Later he shows that points in involution on a line form two superposed projective point-rows.

The seven chapters of 74 pages that have been mentioned constitute Lehmer's course in the essentials of synthetic projective geometry. But this is not all that the book contains. Chapter VII is entitled "Metrical Properties of the Conic Sections." It is here that the author proves that point-rows of the second order are really conic sections. He gets hold of the metrical properties by means of the infinitely distant elements of the plane. The polar line of an infinitely distant point is a diameter, the pole of the infinitely distant line is the center, etc. Point-rows of the second order are classified as hyperbola, parabola and ellipse according as the curve has two points, one point or no point at infinity; and then their analytic geometry equations are found to be the well-known forms $xy = constant$, $y^2 = 2px$, and $x^2/a^2 + y^2/b^2 = 1$. This identifies the point-rows of the second order with the hyperbola, parabola and ellipse as they are known to the student of analytic geometry. If that student had in his analytic

geometry course the proof that these curves are really sections of a circular cone, he is then sure that point-rows of the second order are really conic sections. The reviewer has a preference for another method of proving that point-rows of the second order are the same as conic sections, namely the method given in Cremona's *Projective Geometry*, where it is proved that the locus of the points of intersection of corresponding rays of two projective point-rows is a figure in homology with a circle. Of course this necessitates a chapter on homology which Lehmer's book does not have. If synthetic projective geometry is to be studied by students who have not had analytic geometry, and Lehmer states in the preface his belief that it is "destined shortly to force its way down into the secondary school," it seems to the reviewer that the identifying of point-rows of the second order with conic sections had better be done by means of homology. The book as it stands has too many references to analytic geometry to be suitable for the student who has not had a course in that subject. And if the references to analytic geometry are omitted, the course seems incomplete.

Chapter IX is entitled "Metrical Properties of Involutions." In it are defined axes, foci and directrices of conics, and the well-known focus-directrix property is established for all the conics.

Chapter X, consisting of 23 pages, gives a good account of the historical development of the subject.

For reasons already mentioned it seems to the reviewer that the book is not well adapted for use by those who have not studied analytic geometry. But it can be made the basis of a good semester course for college juniors and seniors if supplemented by lectures on problems of the second degree and perhaps other topics. As it stands it is hardly extensive enough.

One other comment comes to mind. The principle of duality is responsible for much awkward phraseology. Students and writers of textbooks too often write the correlative of a statement by merely putting the word plane for point and point for plane and thereby getting a sentence which is at best an awkward statement of what was intended. The one such example in this book which caught the reviewer's attention is this sentence in paragraph 16 of Chapter I. "The point, considered as made up of all the lines and planes passing through it, is called a point-system." To speak of a point as made up of lines and planes will strike the student as absurd. Mathematical literature has altogether too many such statements that strike the reader as absurd, although the absurdity may be minimized when the reader understands the origin of the phraseology. Lehmer gives an interesting comment along this line in paragraph 181 in the chapter on the history of synthetic geometry: "When the geometer speaks of the two real or imaginary intersections of a straight line with a conic, he is really speaking the language of algebra. *Apart from the algebra involved,* it is the height of absurdity to try to distinguish between the two points in which a line *fails to meet a conic.*

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